

# Dilatonic effects on a falling test mass in scalar-tensor theory

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## Abstract

Effects of a 4d dilaton field on a falling test mass are examined from the Einstein frame perspective of scalar-tensor theory. Results are obtained for the centripetal acceleration of particles in circular orbits, and the radial acceleration for particles with pure radial motion. These results are applied to the specific case of nonrelativistic motion in the weak field approximation of Brans-Dicke theory, employing the exact Xanthopoulos-Zannias solutions. For a given parameter range, the results obtained from Brans-Dicke theory are qualitatively dramatically different from those of general relativity. Comments are made concerning a comparison with the general relativistic results in the limit of an infinite Brans-Dicke parameter.

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## I. INTRODUCTION

Scalar-tensor theories form a class of candidates for a modified description of gravity, and some type of modified gravity at large distances could give rise to observable deviations from general relativity. Brans-Dicke theory[1] is a prototypical scalar-tensor theory where, in a Jordan frame representation, a massless scalar field couples nonminimally to the Ricci curvature scalar. However, more general scalar-tensor theories allow different couplings of the scalar “dilaton” field to the curvature, as well as accommodating nonzero scalar field potentials. Four dimensional scalar-tensor theories arise from a variety of theoretical approaches aimed at achieving unification and/or explaining certain types of observations. Such approaches include Kaluza-Klein type models, string theory, and brane-world models involving extra space dimensions, and result in effective four dimensional models of gravity with a nonminimally coupled scalar field[2]. Therefore, a study of the effects presented by a general form of scalar-tensor theory will include the effects that emerge from a variety of higher dimensional models, as well as four dimensional scalar-tensor theories that may not require extra dimensions.

A fairly general form of a scalar-tensor theory is considered here, and we concentrate on the Einstein frame representation of the theory where the dilatonic effects and the metric tensor field effects can be distinguished more easily. We then proceed to find expressions for the motion of a test particle moving in a static, spherically symmetric background. Expressions are obtained for (1) the angular speed of a test mass in circular motion, and (2) the radial acceleration of a particle undergoing pure radial motion. Simplification results when we consider nonrelativistic motion. As an example, we apply these expressions to the exact analytical vacuum solutions of Brans-Dicke theory[1], i.e., the Xanthopoulos-Zannias solutions[3], which solve the Einstein frame field equations. The differences between the Brans-Dicke results and the general relativity (GR) results are seen, and for a given parameter range, are dramatically different in a qualitative sense. Comments are also offered to illustrate in a concrete way, that, as pointed out by Faraoni[4],[5], when the matter stress-energy vanishes, GR is not generically recovered from the Brans-Dicke theory in the limit of an infinite Brans-Dicke parameter.

## II. CONFORMAL FRAMES

Consider a Jordan frame representation of a scalar-tensor theory of the form

$$S = \int d^4x \sqrt{\tilde{g}} \left\{ \frac{F(\tilde{\phi})}{2\kappa^2} \tilde{R}[\tilde{g}_{\mu\nu}] + \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - V(\tilde{\phi}) \right\} + S_m[\tilde{g}_{\mu\nu}] \quad (2.1)$$

where  $\kappa^2 = 16\pi G$ ,  $\tilde{g} = |\det \tilde{g}_{\mu\nu}|$ , the scalar field  $\tilde{\phi}$  is identified as a 4d dilaton with a potential  $V(\tilde{\phi})$ , and a metric signature  $(+, -, -, -)$  is used. The Jordan frame metric and line element are given by  $d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu$ . The matter action  $S_m[\tilde{g}_{\mu\nu}]$  is constructed from the metric  $\tilde{g}_{\mu\nu}$  and matter terms. For instance, a classical particle action can be written as

$$S_{m,cl} = - \sum_A \int m_{0,A} d\tilde{s}_A = - \sum_A \int m_{0,A} [\tilde{g}_{\mu\nu}(x_A) dx_A^\mu dx_A^\nu]^{1/2} \quad (2.2)$$

where  $m_{0,A}$  is the mass of particle  $A$  in the Jordan frame, assumed to be a constant. A field theoretic matter action is

$$S_m = \int d^4x \sqrt{\tilde{g}} \tilde{\mathcal{L}}_m(\tilde{g}_{\mu\nu}, \psi) \quad (2.3)$$

where  $\psi$  labels matter fields. A classical matter Lagrangian density can be defined by[6],[7]

$$\sqrt{\tilde{g}} \tilde{\mathcal{L}}_{cl} = - \sum_A \int m_{0,A} [\tilde{g}_{\mu\nu}(x_A) dx_A^\mu dx_A^\nu]^{1/2} \delta^{(4)}(x - x_A) \quad (2.4)$$

The associated stress-energy tensors for field theoretic or classical actions

$$\tilde{\mathcal{T}}^{\mu\nu} = \frac{2}{\sqrt{\tilde{g}}} \frac{\partial(\sqrt{\tilde{g}} \tilde{\mathcal{L}}_m)}{\partial \tilde{g}_{\mu\nu}}, \quad \tilde{\mathcal{T}}_{cl}^{\mu\nu} = - \frac{2}{\sqrt{\tilde{g}}} \frac{\partial(\sqrt{\tilde{g}} \tilde{\mathcal{L}}_{cl})}{\partial \tilde{g}_{\mu\nu}} \quad (2.5)$$

then give  $\tilde{\mathcal{T}}_{00} > 0$  in both cases.

The Einstein frame representation of this theory is obtained with a rescaling of the metric and scalar field[8],[9],[10] :

$$\tilde{g}_{\mu\nu} \rightarrow g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}, \quad \Omega = \sqrt{F(\tilde{\phi})}, \quad \tilde{\phi} \rightarrow \phi(\tilde{\phi}), \quad \frac{d\phi}{d\tilde{\phi}} = \frac{1}{F} \left\{ F + \frac{3}{16\pi G} [F'(\tilde{\phi})]^2 \right\}^{1/2} \quad (2.6)$$

where  $F'(\tilde{\phi}) = dF/d\tilde{\phi}$ , giving an Einstein frame representation

$$S = \int d^4x \sqrt{g} \left\{ \frac{1}{2\kappa^2} R[g_{\mu\nu}] + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U(\phi) \right\} + S_m(\Omega^{-2} g_{\mu\nu}) \quad (2.7)$$

The potential  $U(\phi)$  depends upon the functions  $F(\tilde{\phi})$  and  $V(\tilde{\phi}(\phi))$ ,

$$U(\phi) = \frac{V}{\Omega^4} = \frac{V[\tilde{\phi}(\phi)]}{F^2[\tilde{\phi}(\phi)]}$$

(See, for example, [9].) The Einstein frame line element is  $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \Omega^2 d\tilde{s}^2 = F(\tilde{\phi}) d\tilde{s}^2$ . In the Einstein frame a particle has a mass  $m$ , which is generally position dependent

due to its dependence on the scalar field  $\tilde{\phi}$ . Consider, for example, a classical matter action of the form in (2.2),

$$-S_m = \int m_0 d\tilde{s} = \int m_0 (\Omega^{-1} ds) = \int m_0 F^{-1/2} ds = \int m ds \quad (2.8)$$

so that the Einstein frame mass  $m$  is related to the Jordan frame mass  $m_0$  by [8],[9]

$$m = \Omega^{-1} m_0 = F^{-\frac{1}{2}}(\tilde{\phi}) m_0 \quad (2.9)$$

Therefore, a particle having a constant mass  $m_0$  in the Jordan frame will have a mass  $m = F^{-1/2} m_0$  in the Einstein frame. Since the fields  $\tilde{\phi}$  and  $\phi$  generally depend on spacetime position, then the Einstein frame mass  $m = m(x^\mu)$  in general. The matter Lagrangian density in the Einstein frame,  $\mathcal{L}_m$ , is related to that in the Jordan frame,  $\tilde{\mathcal{L}}_m$ , by [8],[9]

$$\mathcal{L}_m = \Omega^{-4} \tilde{\mathcal{L}}_m(\tilde{g}_{\mu\nu}) = F^{-2} \tilde{\mathcal{L}}_m(\tilde{g}_{\mu\nu}) \quad (2.10)$$

A particular example is that of Brans-Dicke (BD) theory [1], with a Jordan frame action ( $G = 1$ )

$$S = \frac{1}{16\pi} \int d^4x \sqrt{\tilde{g}} \left\{ \tilde{\phi} \tilde{R} + \frac{\omega_{BD}}{\tilde{\phi}} \tilde{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} \right\} + S_m(\tilde{g}_{\mu\nu}) \quad (2.11)$$

A conformal transformation to the Einstein frame is given by [11]

$$g_{\mu\nu} = \tilde{\phi} \tilde{g}_{\mu\nu}, \quad g^{\mu\nu} = \tilde{\phi}^{-1} \tilde{g}^{\mu\nu}, \quad \sqrt{g} = \tilde{\phi}^2 \sqrt{\tilde{g}}, \quad \phi = \sqrt{2a} \ln \tilde{\phi}, \quad a = \omega_{BD} + \frac{3}{2} \quad (2.12)$$

and the action in the Einstein frame then takes the form

$$S = \frac{1}{16\pi} \int d^4x \sqrt{g} \left\{ R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\} + S_m(\tilde{\phi}^{-1} g_{\mu\nu}) \quad (2.13)$$

where  $R$  is built from  $g_{\mu\nu}$  and Einstein gravity is coupled to a massless Einstein frame scalar dilaton field  $\phi$ . Using  $g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}$  as in (2.6), we identify  $\Omega = \tilde{\phi}^{1/2}$  and from (2.9) we have

$$m = \Omega^{-1} m_0 = \tilde{\phi}^{-1/2} m_0 \quad (2.14)$$

(The kinetic term in (2.11) is in noncanonical form, but a rescaling of the scalar field [9]  $\tilde{\phi} \rightarrow \bar{\phi}^2/(8\omega_{BD})$  would put the kinetic term into a canonical form as in (2.1), with  $F(\bar{\phi}) \propto \bar{\phi}^2/(8\omega_{BD})$ .) Terms in the matter Lagrangian  $\mathcal{L}_m = \Omega^{-4} \tilde{\mathcal{L}}_m(\tilde{g}_{\mu\nu}) = \tilde{\phi}^{-2} \tilde{\mathcal{L}}_m(\tilde{g}_{\mu\nu})$  pick up an anomalous coupling to the dilaton  $\tilde{\phi}$ .

A classical test particle of mass  $m$  moving in a gravitational field described by  $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$  has an action like that in (2.8),

$$S = - \int m [g_{\mu\nu}u^\mu u^\nu]^{1/2} ds \quad (2.15)$$

where  $u^\alpha = dx^\alpha/ds$  is subject to the “on shell” constraint  $u_\alpha u^\alpha = 1$ . The “geodesic” equation of the motion (in an otherwise matter-free region) obtained from (2.15) can be written in the form[8]

$$\frac{d}{ds} (m g_{\mu\nu} u^\nu) - \frac{1}{2} m (\partial_\mu g_{\alpha\beta}) u^\alpha u^\beta - \partial_\mu m = 0 \quad (2.16)$$

or in the form

$$\frac{du^\nu}{ds} = -\Gamma_{\alpha\beta}^\nu u^\alpha u^\beta + \partial_\mu (\ln m) (g^{\mu\nu} - u^\mu u^\nu) \quad (2.17)$$

The first term on the right hand side of (2.17) is recognized as the gravitational acceleration due to the metric field  $g_{\mu\nu}$ , while the second term on the right hand side represents the dilatonic acceleration due to the scalar field, and therefore a deviation from pure, unforced, geodesic motion. Since  $m(x^\mu) \propto \Omega^{-1}(x^\mu) = F^{-1/2}(x^\mu)$ , the motion of a particle in the Einstein frame of a scalar-tensor theory where  $\partial_\mu m \neq 0$ , will differ from that described by general relativity (GR) where  $m = \text{const}$ . This reflects the fact that the Jordan frame metric  $\tilde{g}_{\mu\nu}$  for a scalar-tensor theory will generally be different from the metric of GR. Since the acceleration of a test mass in the Einstein frame depends upon the tensor field  $g_{\mu\nu}$  as well as the dilatonic acceleration due to the scalar field  $\phi$ , it is not enough to consider the asymptotic form of the metric alone, e.g.,  $g_{00} \rightarrow 1$ , for the case of an asymptotically flat spacetime.

### III. MOTION IN A STATIC, SPHERICALLY SYMMETRIC BACKGROUND

We now focus upon the motion of a classical test particle of mass  $m(r)$  moving under the influence of a metric field  $g_{\mu\nu}$  in the Einstein frame of a scalar-tensor theory that can be written in the form of eq.(2.1). We assume that  $g_{\mu\nu}$  and  $m$  are static and spherically symmetric functions, independent of  $t$  and azimuth angle  $\varphi$ , with  $g_{\mu\nu}$  being diagonal, and consider motion in the equatorial plane,  $\theta = \pi/2$ . The special cases of circular motion and pure radial motion will be considered by using (2.16) or (2.17), along with the constraint  $u_\alpha u^\alpha = 1$ . Different coordinate systems can be used (Schwarzschild-like or isotropic), but we take the metric to have a general form

$$ds^2 = e^{f(r)} dt^2 - e^{-h(r)} dr^2 - \rho(r) r^2 d\Omega^2 \quad (3.1)$$

where  $\rho(r) = e^{-h(r)}$  for isotropic coordinates, and  $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ .

First, we point out that if the motion is initially within the equatorial plane  $\theta = \pi/2$ , then it remains in this plane, so that  $u^\theta = d\theta/ds = 0$ . This is seen from the  $\theta$  component of (2.16), which reduces to  $\frac{d(mu_\theta)}{ds} - \frac{m}{2}\rho(r)r^2 \sin\theta \cos\theta (u^\varphi)^2 = 0$ , so that if  $\theta = \pi/2$  and  $u_\theta = 0$  initially, then  $d(mu_\theta)/ds = 0$  initially (no  $\theta$  component of acceleration), so that motion remains in the  $\theta = \pi/2$  plane.

For the  $t$  component equation,  $\partial_0 g_{\alpha\beta} = 0$  and  $\partial_0 m = 0$ , so  $\frac{d}{ds}(mu_0) = 0$ , which gives

$$p_0 = mu_0 = E; \quad u^0 = \frac{E}{mg_{00}} \quad (3.2)$$

where the energy  $E$  is a constant parametrizing the particular orbit. For example, a given circular orbit has a fixed value of  $E$ , but this value will generally depend upon the orbital radius  $r$ , so that  $E = E(r)$  is a constant on the orbit. For pure radial motion,  $E$  might characterize the asymptotic energy of the test mass, and different values of  $E$  characterize different radial orbits (e.g., different turning points).

Similarly, for the  $\varphi$  equation,  $\frac{d}{ds}(mu_\varphi) = 0$ , with

$$p_\varphi = mu_\varphi = mg_{\varphi\varphi}u^\varphi = -L; \quad u_\varphi = \frac{-L}{m}, \quad u^\varphi = \frac{-L}{mg_{\varphi\varphi}} \quad (3.3)$$

where the angular momentum  $L$  is a constant of motion characterizing the orbit, and  $g_{\varphi\varphi}$  is evaluated at  $\theta = \pi/2$  for circular orbits. For pure radial motion,  $L = 0$ , but for a circular orbit  $L$  depends upon the orbital radius, as with Newtonian gravity.

The radial equation reduces to

$$\frac{d}{ds}(mu_r) - \frac{1}{2}m [(\partial_r g_{00})(u^0)^2 + (\partial_r g_{rr})(u^r)^2 + (\partial_r g_{\varphi\varphi})(u^\varphi)^2] - \partial_r m = 0 \quad (3.4)$$

and the constraint equation is  $u_\alpha u^\alpha = g^{00}(u_0)^2 + g^{\varphi\varphi}(u_\varphi)^2 + g_{rr}(u^r)^2 = 1$ . Using (3.2) and (3.3), this constraint gives

$$(u^r)^2 = \frac{1}{|g_{rr}|} \left[ \frac{E^2}{g_{00}m^2} - \frac{L^2}{|g_{\varphi\varphi}|m^2} - 1 \right] \quad (3.5)$$

The kinematically allowed regions where the test mass can propagate are defined by  $(u^r)^2 \geq 0$ , with radial turning points given by  $(u^r)^2 = 0$ .

We could also define an effective potential  $\mathcal{V}$  for the radial motion by (see, for example, [12], [13])  $(u^r)^2 + \mathcal{V}^2 = E^2/m^2$ , where

$$\mathcal{V}^2 = \frac{1}{|g_{rr}|} \left( 1 + \frac{L^2}{|g_{\varphi\varphi}|m^2} \right) + \frac{E^2}{m^2} \left( 1 - \frac{1}{g_{00}|g_{rr}|} \right) \quad (3.6)$$

with radial turning points determined by  $E/m = \mathcal{V}$ .

### A. Circular motion

For circular motion  $u^r = 0$  and  $L$  and  $E$  are constants of the particular orbit. We write

$$u^\varphi = \frac{-L}{mg_{\varphi\varphi}} = \frac{d\varphi}{ds} = u^0 \frac{d\varphi}{dt} = u^0 \omega = \frac{E}{mg_{00}} \omega \quad (3.7)$$

where the angular speed  $\omega = d\varphi/dt$ , and we have used (3.2) and (3.3). Therefore the angular speed  $\omega$  can be written as

$$\omega = \frac{L}{E} \frac{g_{00}}{|g_{\varphi\varphi}|} \quad (3.8)$$

with  $E$  and  $L$  related by (3.5) with  $u^r$  set to zero. For example, consider the case of nonrelativistic circular motion of a test particle with constant mass  $m$  due to Newtonian gravity in a Minkowski spacetime, where  $g_{00} \rightarrow 1$  and  $|g_{\varphi\varphi}| \rightarrow r^2$  in the equatorial plane, and  $E \rightarrow m$ . We then have from (3.8)  $\omega = L/mr^2$ , the ordinary Newtonian relation.

Now the relation between  $E$  and  $L$ , given by (3.5) with  $u^r$  set to zero, is equivalent to the constraint equation  $p_\mu p^\mu = m^2$  with  $p^r = 0$ , and leads to

$$\frac{E}{m} = \sqrt{g_{00}} \left[ 1 + \frac{L^2}{|g_{\varphi\varphi}| m^2} \right]^{1/2} \quad (3.9)$$

The second term within brackets on the right is recognized as an orbital kinetic energy (per unit mass) term. Eq. (3.9) allows  $E$  to be eliminated from the expression in (3.8), leaving  $\omega$  to be determined by  $L$ , along with the metric  $g_{\mu\nu}$  and mass function  $m(r)$ . The radial equation (3.4),

$$(\partial_r g_{00}) \left( \frac{E}{mg_{00}} \right)^2 + (\partial_r g_{\varphi\varphi}) \left( \frac{L}{mg_{\varphi\varphi}} \right)^2 = -2 \frac{\partial_r m}{m} \quad (3.10)$$

then allows a determination of  $L$  in terms of the  $g_{\mu\nu}$  and  $m$ . Thus  $\omega$ , evaluated on the orbit with radius  $r$ , will ultimately depend not only upon the metric  $g_{\mu\nu}$ , but also upon the mass function  $m(r)$  and its rate of change  $\partial_r m$ , evaluated on the orbit with radius  $r$ .

#### 1. *Nonrelativistic limit:* $|\vec{p}|/m \ll 1$ :

The above procedure and eq.(3.8) simplifies in the nonrelativistic limit of low velocities,  $v \ll 1$ , or where the particle kinetic energy is much smaller than its mass energy,  $|p_i p^i| \ll m^2$ . Then  $p_\mu p^\mu = g^{00} E^2 + p_i p^i = m^2 \approx g^{00} E^2$ , or

$$E \approx m \sqrt{g_{00}} \quad (3.11)$$

(This is also obtained from (3.9) when we drop the orbital kinetic energy term.)

The radial equation (3.10), with the use of (3.11) and some rearrangement, yields

$$\frac{L}{m} \approx |g_{\varphi\varphi}| \left\{ \frac{1}{\partial_r |g_{\varphi\varphi}|} \partial_r [\ln(m^2 g_{00})] \right\}^{1/2} \quad (3.12)$$

in the nonrelativistic limit. Using (3.11) and (3.12), the angular speed in (3.8) is given in the nonrelativistic limit by

$$\omega^2 \approx \frac{\partial_r(m^2 g_{00})}{m^2 \partial_r |g_{\varphi\varphi}|} \quad (3.13)$$

To test this, we use the Schwarzschild solution in Schwarzschild coordinates, with  $m = \text{const}$ ,  $g_{00} = 1 - 2GM/r$ , and  $|g_{\varphi\varphi}| = r^2$  in the equatorial plane. Eq.(3.13) then gives the Newtonian limit,  $\omega^2 = GM/r^3$  and a centripetal acceleration  $|a_c| = \omega^2 r = GM/r^2$ , i.e., the ordinary Newtonian gravitational field produced by a static, spherically symmetric body of mass  $M$ . However, it is possible that a scalar-tensor theory with nonconstant mass and a different metric field could yield a dramatically different result. An example is provided later for Brans-Dicke theory.

## B. Radial Motion

We take  $\theta = \pi/2$ ,  $\varphi = \text{const}$ , so that  $u^\theta = u^\varphi = 0$  for pure radial motion, and the constraint equation becomes  $g^{00}(u_0)^2 - |g_{rr}|(u^r)^2 = 1$ , with  $u_0 = E/m$ . We also have the radial component of equation (2.17), which becomes

$$\frac{du^r}{ds} = - [\Gamma_{00}^r (u^0)^2 + \Gamma_{rr}^r (u^r)^2] + \frac{\partial_r m}{m} [g^{rr} - (u^r)^2] \quad (3.14)$$

with  $\Gamma_{00}^r = -\frac{1}{2}g^{rr}\partial_r g_{00}$  and  $\Gamma_{rr}^r = \frac{1}{2}g^{rr}\partial_r g_{rr}$ . The constraint equation

$$(u^r)^2 = \frac{1}{|g_{rr}|} \left[ \frac{E^2}{g_{00}m^2} - 1 \right] \quad (3.15)$$

and  $u^0 = g^{00}E/m$  can be used in (3.14), allowing the proper radial acceleration to be given by

$$g_{rr} \frac{du^r}{ds} = \frac{1}{2}(\partial_r g_{00}) \left[ \frac{E^2}{g_{00}^2 m^2} \right] - \frac{1}{2}g^{rr}(\partial_r g_{rr}) \left[ 1 - \frac{E^2}{g_{00}m^2} \right] + \frac{\partial_r m}{m} \left[ \frac{E^2}{g_{00}m^2} \right] \quad (3.16)$$



1. *Nonrelativistic limit:*  $|\vec{p}|/m \ll 1$ :

In the nonrelativistic limit  $|p_r p^r|/m^2 \ll 1$ , or  $g_{rr}(u^r)^2 \ll 1$ , then (3.15) implies that  $\frac{E^2}{g_{00}m^2} \approx 1$ , i.e., the condition given by (3.11), and we also have  $ds \approx \sqrt{g_{00}}dt$ . The geodesic equation then takes the simplified form

$$\frac{d^2 r}{dt^2} \approx \left( \frac{g_{00}}{g_{rr}} \right) \left[ \frac{1}{2}(\partial_r g_{00}) + \frac{\partial_r m}{m} \right] \quad (3.17)$$

in describing the radial acceleration of a test mass  $m$  in the nonrelativistic limit, where for a scalar-tensor theory  $m = m(r)$  in the Einstein frame. As an example, we again apply this to the Schwarzschild case, where  $m$  is constant and  $g_{00} = -g_{rr}^{-1} = (1 - 2GM/r)$ , to get a radial acceleration  $a_r = a_c = -GM/r^2$ , the usual Newtonian limit. However, we will also consider an example from Brans-Dicke theory.

#### IV. APPLICATION TO BRANS-DICKE THEORY

We now apply the results of (3.13) and (3.17) to the case of a static, spherically symmetric background of Brans-Dicke (BD) theory. The Jordan frame representation of the BD theory is given by (2.11). Exact static, spherically symmetric vacuum solutions in the Jordan frame were provided by Brans[14]. The conformal transformations described by (2.12) allows the theory to be rewritten in the Einstein frame representation, given by (2.13). The BD vacuum solutions in the Einstein frame, as well as the higher dimensional generalizations, have been provided by Xanthopoulos and Zannias[3]. Cai and Myung[11] have also studied these solutions, explicitly relating the Jordan frame solutions and the Einstein frame solutions through the transformations of (2.12). We apply these solutions to describe the region exterior to some neutral, nonrotating astrophysical object of BD theory, and look at the asymptotic limit  $r \gg r_0$ . (There is a naked singularity at  $r = r_0$ , except in the case of the Schwarzschild limit, where the solution coincides with the Schwarzschild solution[3],[11].) However, the solution inside the astrophysical object will not be a vacuum solution, so that we do not generally expect a physical singularity to exist. For an astrophysical object like a star or planet, we expect that  $r/r_0 \gg 1$  for all regions outside the surface.

The static neutral solutions, with isotropic coordinates, are presented here for the special 4d case:

$$ds^2 = e^f dt^2 - e^{-h}(dr^2 + r^2 d\Omega^2) \quad (4.1)$$

$$e^f = g_{00} = \xi^{2\gamma}; \quad \xi = \left( \frac{r - r_0}{r + r_0} \right) \quad (4.2)$$

$$e^{-h} = |g_{rr}| = \left(1 - \frac{r_0^2}{r^2}\right)^2 \xi^{-2\gamma} = e^{-f} \left(1 - \frac{r_0^2}{r^2}\right)^2 \quad (4.3)$$

$$\phi = \pm \tilde{\gamma} \ln \xi = \sqrt{2a} \ln \tilde{\phi}; \quad \tilde{\gamma} = [4(1 - \gamma^2)]^{1/2} \quad (4.4a)$$

$$\tilde{\phi} = \xi^\Gamma; \quad \Gamma = \pm \frac{\tilde{\gamma}}{\sqrt{2a}} = \pm \left[ \frac{2}{a}(1 - \gamma^2) \right]^{1/2} = \pm |\Gamma| \quad (4.4b)$$

where  $r_0$  and  $\gamma$  are integration constants ( $r_0 > 0$ ), and we have defined

$$\xi = \left( \frac{r - r_0}{r + r_0} \right) \leq 1, \quad \tilde{\gamma} = [4(1 - \gamma^2)]^{1/2}, \quad \Gamma = \pm \frac{\tilde{\gamma}}{\sqrt{2a}} = \pm \left[ \frac{2}{a}(1 - \gamma^2) \right]^{1/2} \quad (4.5)$$

These are the Einstein frame fields and solutions, with  $0 \leq \gamma \leq 1$  for the description of physical (nonnegative ADM mass) solutions. There is a naked singularity at  $r = r_0$  where  $R = g^{\mu\nu} R_{\mu\nu} \rightarrow \infty$  unless  $\gamma = 1$  and  $\phi = 0$  (the Schwarzschild solution).

*Note:* In the set of solutions presented in ref.[3], only the solution with the + sign in (4.4a), i.e.,  $\phi = +\tilde{\gamma} \ln \xi$ , is presented. However, the second solution  $\phi = -\tilde{\gamma} \ln \xi$  is seen to exist due to the invariance of the action and equations of motion (EoM) under the transformations  $g_{\mu\nu} \rightarrow g_{\mu\nu}$ ,  $\phi \rightarrow -\phi$ . Thus if  $\phi$  is a solution to the EoM, then so is  $-\phi$  (see, for example, refs.[11] and[15]). Therefore  $\phi$  can be positive or negative, and the Brans-Dicke scalar  $\tilde{\phi} = \xi^\Gamma = \xi^{\pm|\Gamma|}$  can be either a decreasing or an increasing function of  $r$  and  $\xi$ . The Einstein frame mass  $m$  of a test particle is given by (2.14) and (4.4b),

$$m = m_0 \tilde{\phi}^{-1/2} = m_0 \xi^{-\Gamma/2} \quad (4.6)$$

where  $m_0$  is the constant Jordan frame mass.

We now want to consider the asymptotic forms of these solutions for which  $r_0/r \ll 1$ . In this case we have the following approximations to  $O(r_0/r)$ .

$$\begin{aligned} \xi &\approx 1 - 2\frac{r_0}{r}, & g_{00} &\approx 1 - 4\gamma\frac{r_0}{r}, & |g_{rr}| &\approx 1/g_{00}, \\ |g_{\varphi\varphi}|_{\theta=\pi/2} &\approx r^2 \left(1 + 4\gamma\frac{r_0}{r}\right), & \partial_r |g_{\varphi\varphi}|_{\theta=\pi/2} &\approx 2r \left(1 + 2\gamma\frac{r_0}{r}\right), \\ \frac{m^2}{m_0^2} &\approx \left(1 + 2\Gamma\frac{r_0}{r}\right), & \frac{m^2}{m_0^2} g_{00} &\approx 1 - 2(2\gamma - \Gamma)\frac{r_0}{r}, \end{aligned} \quad (4.7)$$

For the case of nonrelativistic particle motion, applying (4.7) to (3.13) for the case of circular motion yields the result

$$\omega^2 \approx (2\gamma - \Gamma) \frac{r_0}{r^3}, \quad a_c = -\omega^2 r \approx -(2\gamma - \Gamma) \frac{r_0}{r^2} \quad (4.8)$$

The Schwarzschild case is obtained for  $\gamma = 1$ ,  $\Gamma = 0$ , and the identification  $r_0 = GM/2$ , where  $M$  is the mass of the gravitating object[3],[12]. The Schwarzschild limit therefore gives  $\omega^2 \rightarrow GM/r^3$  and  $a_c \rightarrow -GM/r^2$ , i.e., the Newtonian limit of the gravitational field far from the Schwarzschild radius. (The Schwarzschild radial coordinate  $R$  is related to the isotropic coordinate  $r$  by[3],[12]  $R = r(1 + r_0/r)^2$ , with  $R \rightarrow r$  asymptotically.) Similarly, applying (4.7) to (3.17) for the case of radial motion, we have

$$\frac{d^2r}{dt^2} \approx -(2\gamma - \Gamma) \frac{r_0}{r^2} \quad (4.9)$$

We therefore have the same gravitational acceleration,  $a_c = a_r$ , for circular or radial motion, with the expected Newtonian limit for the Schwarzschild case.

A qualitative distinction between GR and BD in the weak field limit is seen for the case when  $\Gamma > 2\gamma$ , in which case the radial acceleration  $a_r \approx (\Gamma - 2\gamma)r_0/r^2$  given by (4.9), becomes *positive* rather than *negative*, indicating a *repulsion* rather than an *attraction*. Similarly, (4.8) implies that  $\omega^2 < 0$  in this case, i.e., circular orbits do not exist, implying a *repulsion*, rather than *attraction*. This is seen as an example where there is a dilatonic repulsion that dominates the metric field attraction, since, from (4.7), the metric field produces a  $g_{00} - 1 < 0$ , and  $g_{00}(\partial_r g_{00}) > 0$ , but  $m(r)$  is a decreasing function with  $\partial_r m < 0$ . Specifically, from (3.17) and (4.7),

$$\frac{d^2r}{dt^2} \approx - \left( \frac{g_{00}}{|g_{rr}|} \right) \left[ \frac{1}{2}(\partial_r g_{00}) + \frac{\partial_r m}{m} \right] \approx (g_{00})^2 \left[ - \left( 2\gamma \frac{r_0}{r^2} \right) + \frac{|\partial_r m|}{m} \right], \quad (\Gamma > 2\gamma) \quad (4.10)$$

showing that the metric field (due to the first term on the right) produces a negative acceleration, but the dilatonic acceleration due to the mass (due to the second term on the right) produces a positive radial acceleration, which overwhelms the negative metric contribution. (An interesting occurrence of repulsive gravity in GR has also been reported[13].)

For the case  $\Gamma = 2\gamma$ , then  $\omega^2 \rightarrow 0$  and  $a_r \rightarrow 0$ , so that a test mass at rest remains at rest. However, for the parameter range  $\Gamma < 2\gamma$ , which is satisfied for  $\Gamma > 0$  with  $|\Gamma| < 2\gamma$ , and for all  $\Gamma < 0$ , the radial acceleration is negative, with an overall attraction.

The solar system constraint on the massless Brans-Dicke theory requires  $\omega_{BD} > 40,000$  [16, 17]. However, Perivolaropoulos[17] has recently re-examined this constraint for the case where the Brans-Dicke scalar has an arbitrary nonzero mass  $m_{BD}$ . The conclusion reached is that for a mass  $m_{BD} \gtrsim 200 \times 10^{-27}$  GeV, all values of  $\omega_{BD} \geq -3/2$  are allowed by solar system observations. (The extent of the deformation of the Xanthopoulos-Zannias solutions due to a tiny scalar mass, however, is not apparent.)

It can also be pointed out that in the limit of  $\omega_{BD} \rightarrow \infty$ , in which case  $a \rightarrow \infty$ ,  $\Gamma \rightarrow 0$ , and the scalar field is removed from the theory ( $\phi \rightarrow 0$  with  $\tilde{\phi} \rightarrow 1$ ), the radial acceleration obtained from the BD theory becomes  $a_r \approx -2\gamma r_0/r^2$ , which is a factor of  $\gamma$  times the Schwarzschild value obtained from GR. The reason for this can be seen from the Xanthopoulos-Zannias solutions (4.1) - (4.4), noting that the metric  $g_{\mu\nu}$  in this case does not collapse to the Schwarzschild metric for  $\gamma \neq 1$  [5]. This illustrates in a concrete way the point made by Faraoni[4],[5] that BD theory does not always reduce to GR in the  $\omega_{BD} \rightarrow \infty$  limit when the matter stress-energy vanishes, with  $\mathcal{T}^{\mu\nu} = 0$ .

## V. SUMMARY

A fairly general form of scalar-tensor theory has been considered, with a focus on the Einstein frame representation of the theory, where scalar field dilatonic effects and metric tensor field effects become distinguishable. Expressions for the motion of a test particle moving in a static, spherically symmetric background are found (1) for the case of circular motion, and (2) for the case of pure radial motion. Simplified expressions are obtained for nonrelativistic particle motion. As an example, these expressions have been applied to the exact analytical vacuum solutions to Brans-Dicke theory, by using the Xanthopoulos-Zannias solutions for the field equations in the Einstein frame. The differences between the Brans-Dicke results and the general relativity results are seen. For a given parameter range, namely, for  $\Gamma > 2\gamma$ , these are dramatically different qualitatively, as the dilatonic repulsion of a test mass is greater than the gravitational attraction due to the tensor field. Furthermore, it is illustrated in a concrete way that, as pointed out previously by Faraoni[4],[5], when the matter stress-energy vanishes,  $\mathcal{T}^{\mu\nu} = 0$ , GR is not automatically recovered from the Brans-Dicke theory in the limit of an infinite Brans-Dicke parameter,  $\omega_{BD} \rightarrow \infty$ . The Xanthopoulos-Zannias solutions in this limit do not coincide with the Schwarzschild solution, unless the Xanthopoulos-Zannias parameter is unity,  $\gamma = 1$ .

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